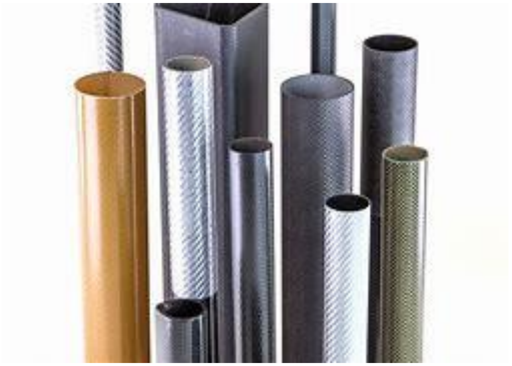
Composite Materials, PhD



Week 3

Macromechanical Analysis of a Lamina

Part 2: 2D Stiffness and Compliance Matrix for Unidirectional Lamina

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Introduction

- • Overview of Macromechanical Analysis of a Lamina
- • Importance of Compliance and Stiffness Matrices
- • Applications in Composite Materials

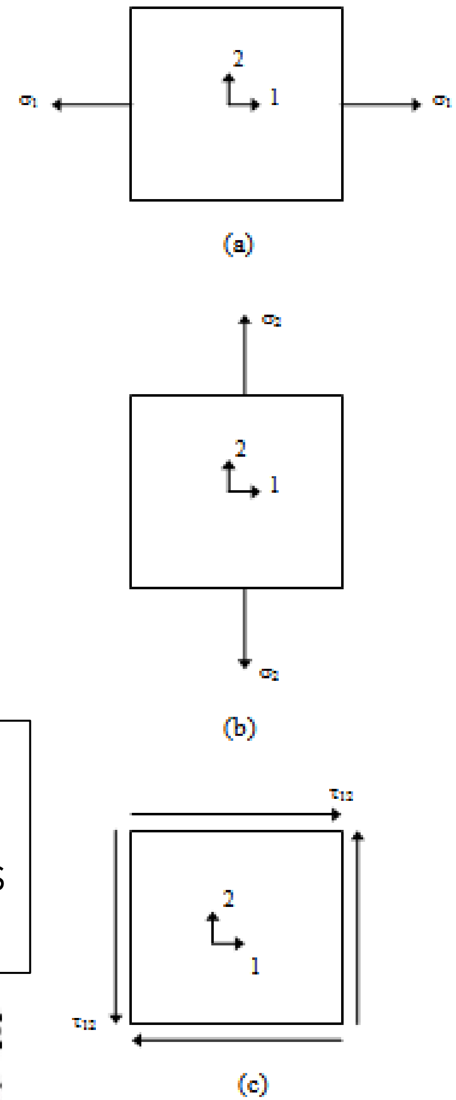
Topics to be Covered

- • Compliance and Stiffness Matrix
- • Transformation of Stresses and Strains
- • Engineering Constants
- • Strength Failure Theories
- • Example Calculations

Objectives

- • Understand 2D Stiffness and Compliance Matrices
- • Apply Hooke's Law for 2D Angle Lamina
- • Analyze Global and Local Stress-Strain Relationships
- • Evaluate Strength Failure Theories

Compliance and Stiffness Matrix Elements in Terms of Elastic Constants



(a) Tensile load in direction 1

(b) Tensile load in direction in 2

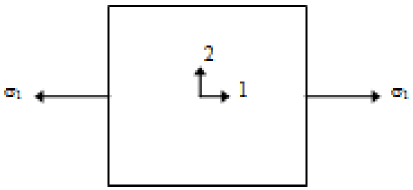
(c) Pure shear stress in plane of 1-2

FIGURE 3.1
Application of stresses to find engineering constants of a unidirectional lamina

Pure Axial Load in Direction 1

Apply a pure axial load in direction 1

$$\sigma_1 \neq 0,\ \sigma_2 = 0,\ \tau_{12} = 0$$



$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} \longrightarrow \begin{array}{c} \varepsilon_1 = S_{11}\sigma_1 \\ \varepsilon_2 = S_{12}\sigma_1 \\ \gamma_{12} = 0 \end{array}$$

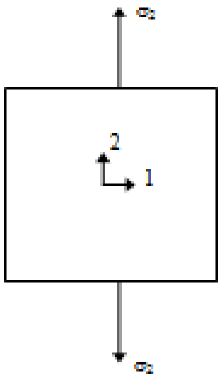
$$E_1 \equiv \frac{\sigma_1}{\varepsilon_1} = \frac{1}{S_{11}} \longrightarrow S_{11} = \frac{1}{E_1}$$

$$\nu_{12} \equiv -\frac{\varepsilon_2}{\varepsilon_1} = -\frac{S_{12}}{S_{11}}. \longrightarrow S_{12} = -\frac{\nu_{12}}{E_1}$$

Pure Axial Load in Direction 2

Apply a pure axial load in direction 2

$\sigma_1 = 0,\ \sigma_2 \neq 0,\ \tau_{12} = 0$



$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} \longrightarrow \begin{matrix} \varepsilon_1 = S_{12}\sigma_2 \\ \varepsilon_2 = S_{22}\sigma_2 \\ \gamma_{12} = 0 \end{matrix}$$

$$E_2 \equiv \frac{\sigma_2}{\varepsilon_2} = \frac{1}{S_{22}} \longrightarrow S_{22} = \frac{1}{E_2}$$

$$\nu_{21} \equiv -\frac{\varepsilon_1}{\varepsilon_2} = -\frac{S_{12}}{S_{22}}. \longrightarrow S_{12} = -\frac{\nu_{21}}{E_2}$$

Reciprocal Relationship

$$S_{12} = -\frac{v_{12}}{E_1}$$

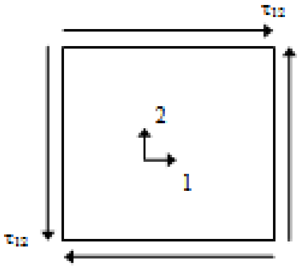
$$S_{12} = -\frac{v_{21}}{E_2}$$

$$\frac{v_{12}}{E_1} = \frac{v_{21}}{E_2}$$

Pure Shear Load in Plane 12

Apply a pure shear load in Plane 12

$$\sigma_1 = 0,\ \sigma_2 = 0,\ \tau_{12} \neq 0$$



$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} \longrightarrow \begin{array}{l} \varepsilon_1 = 0 \\ \varepsilon_2 = 0 \\ \gamma_{12} = S_{66}\tau_{12} \end{array}$$

$$G_{12} \equiv \frac{\tau_{12}}{\gamma_{12}} = \frac{1}{S_{66}} \longrightarrow S_{66} = \frac{1}{G_{12}}$$

Compliance Matrix

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{v_{12}}{E_1} & 0 \\ -\frac{v_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$

Coefficients of Stiffness Matrix

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix}$$

$$Q_{11} = \frac{S_{22}}{S_{11}S_{22} - S_{12}^2}$$

$$Q_{12} = -\frac{S_{12}}{S_{11}S_{22} - S_{12}^2}$$

$$Q_{22} = \frac{S_{11}}{S_{11}S_{22} - S_{12}^2}$$

$$Q_{66} = \frac{1}{S_{66}}$$

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} \frac{E_1}{1 - \nu_{21}\nu_{12}} & \frac{\nu_{12} E_2}{1 - \nu_{21}\nu_{12}} & 0 \\ \frac{\nu_{12} E_2}{1 - \nu_{21}\nu_{12}} & \frac{E_2}{1 - \nu_{21}\nu_{12}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix}$$

3D and 2D compliance matrices

$$\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \gamma_{23}\\ \gamma_{31}\\ \gamma_{12}\end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{13}}{E_1} & 0 & 0 & 0\\ -\frac{\nu_{21}}{E_2} & \frac{1}{E_2} & -\frac{\nu_{23}}{E_2} & 0 & 0 & 0\\ -\frac{\nu_{31}}{E_3} & -\frac{\nu_{32}}{E_3} & \frac{1}{E_3} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{G_{31}} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \tau_{23}\\ \tau_{31}\\ \tau_{12}\end{bmatrix} \longrightarrow \begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & 0\\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0\\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{bmatrix}$$

3D and 2D stiffness matrices

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \tau_{23}\\ \tau_{31}\\ \tau_{12}\end{bmatrix}=\begin{bmatrix}\frac{1-\nu_{23}\nu_{32}}{E_2E_3\Delta} & \frac{\nu_{21}+\nu_{23}\nu_{31}}{E_2E_3\Delta} & \frac{\nu_{31}+\nu_{21}\nu_{32}}{E_2E_3\Delta} & 0 & 0 & 0\\ \frac{\nu_{21}+\nu_{23}\nu_{31}}{E_2E_3\Delta} & \frac{1-\nu_{13}\nu_{31}}{E_1E_3\Delta} & \frac{\nu_{32}+\nu_{12}\nu_{31}}{E_1E_3\Delta} & 0 & 0 & 0\\ \frac{\nu_{31}+\nu_{21}\nu_{32}}{E_2E_3\Delta} & \frac{\nu_{32}+\nu_{12}\nu_{31}}{E_1E_3\Delta} & \frac{1-\nu_{12}\nu_{21}}{E_1E_2\Delta} & 0 & 0 & 0\\ 0 & 0 & 0 & G_{23} & 0 & 0\\ 0 & 0 & 0 & 0 & G_{31} & 0\\ 0 & 0 & 0 & 0 & 0 & G_{12}\end{bmatrix}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \gamma_{23}\\ \gamma_{31}\\ \gamma_{12}\end{bmatrix}$$

$$\longrightarrow$$

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{bmatrix}=\begin{bmatrix}\frac{E_1}{1-\nu_{21}\nu_{12}} & \frac{\nu_{12}E_2}{1-\nu_{21}\nu_{12}} & 0\\ \frac{\nu_{12}E_2}{1-\nu_{21}\nu_{12}} & \frac{E_2}{1-\nu_{21}\nu_{12}} & 0\\ 0 & 0 & G_{12}\end{bmatrix}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{bmatrix}$$

Hooke's Law for a 2D Angle Lamina

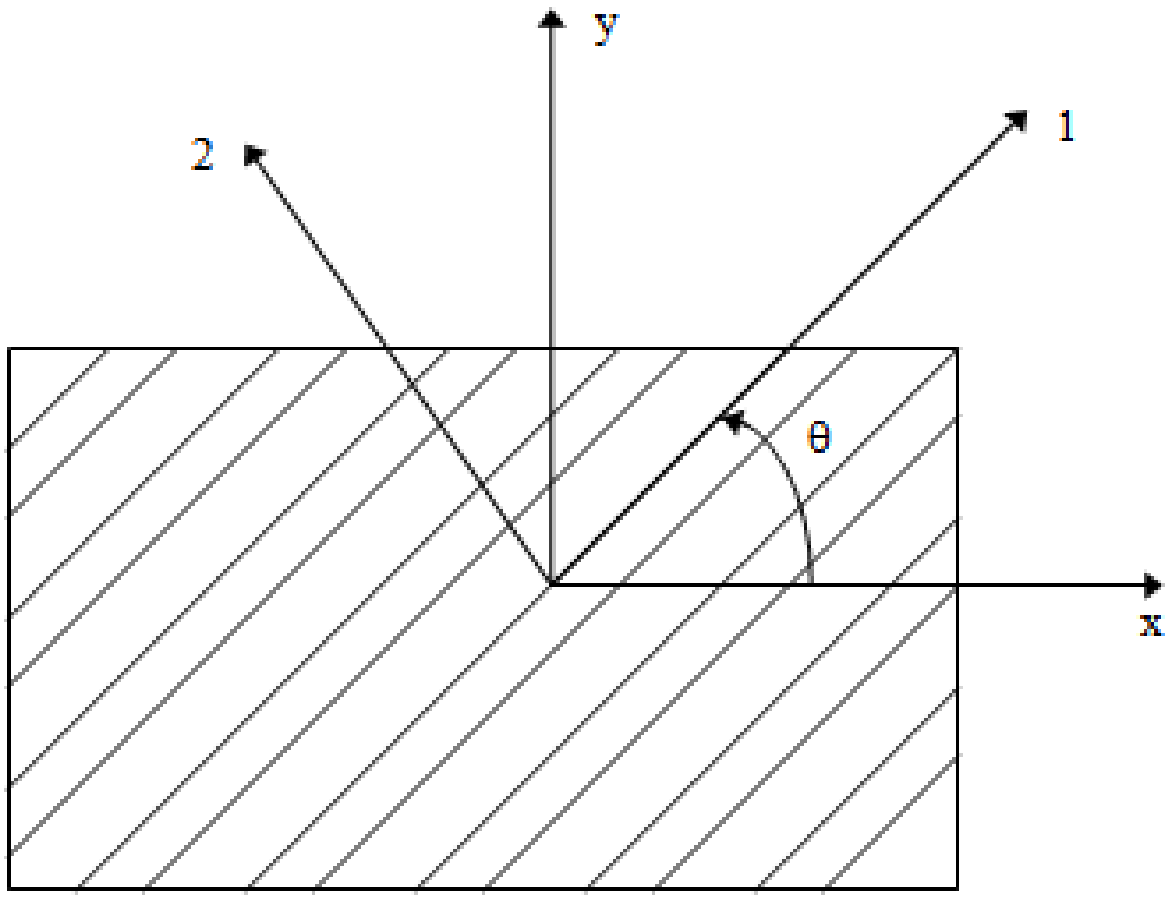


FIGURE 3.2
Local and global axes of an angle lamina

Relationship of Global and Local Stresses

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$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = [T] \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & -2sc \\ s^2 & c^2 & 2sc \\ sc & -sc & c^2 - s^2 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = [T]^{-1} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$

Relationship of Global and Local Strains

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & sc \\ s^2 & c^2 & -sc \\ -2sc & 2sc & c^2 - s^2 \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix}$$

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12}/2 \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy}/2 \end{bmatrix}$$

Relationship of Global and Local Strains

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12}/2 \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2-s^2 \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy}/2 \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy}/2 \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & -2sc \\ s^2 & c^2 & 2sc \\ sc & -sc & c^2-s^2 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12}/2 \end{bmatrix}$$

$$\Rightarrow$$

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12}/2 \end{bmatrix} = [T] \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy}/2 \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy}/2 \end{bmatrix} = [T]^{-1} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12}/2 \end{bmatrix}$$

Expanding Global Strain-Local Strain Relationship

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = [T]^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} [T]^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = [R][T]^{-1}[R]^{-1} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix}$$

Global Stress and Strain

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = [R][T]^{-1}[R]^{-1} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{bmatrix} \longrightarrow \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = [R][T]^{-1}[R]^{-1}[S] \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = [R][T]^{-1}[R]^{-1}[S]\,[T] \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \longrightarrow \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

Transformed Compliance Matrix

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

$$\bar{S}_{11} = S_{11}c^4 + (2S_{12} + S_{66})s^2c^2 + S_{22}s^4,$$

$$\bar{S}_{12} = S_{12}(s^4 + c^4) + (S_{11} + S_{22} - S_{66})s^2c^2,$$

$$\bar{S}_{22} = S_{11}s^4 + (2S_{12} + S_{66})s^2c^2 + S_{22}c^4,$$

$$\bar{S}_{16} = (2S_{11} - 2S_{12} - S_{66})sc^3 - (2S_{22} - 2S_{12} - S_{66})s^3c,$$

$$\bar{S}_{26} = (2S_{11} - 2S_{12} - S_{66})s^3c - (2S_{22} - 2S_{12} - S_{66})sc^3,$$

$$\bar{S}_{66} = 2(2S_{11} + 2S_{22} - 4S_{12} - S_{66})s^2c^2 + S_{66}(s^4 + c^4)$$

Transformed Reduced Stiffness Matrix

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

$$\overline{Q}_{11} = Q_{11}c^4 + Q_{22}s^4 + 2(Q_{12} + 2Q_{66})s^2c^2,$$

$$\overline{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})s^2c^2 + Q_{12}(c^4 + s^4),$$

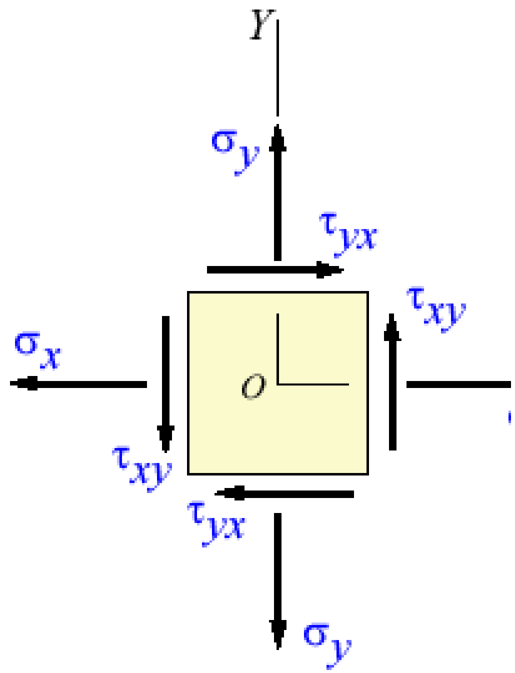
$$\overline{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66})c^3s - (Q_{22} - Q_{12} - 2Q_{66})s^3c,$$

$$\overline{Q}_{22} = Q_{11}s^4 + Q_{22}c^4 + 2(Q_{12} + 2Q_{66})s^2c^2,$$

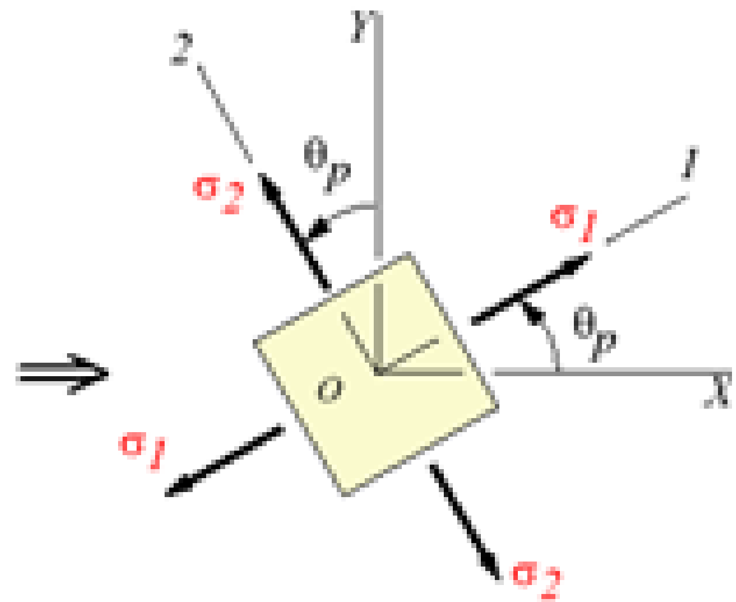
$$\overline{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66})cs^3 - (Q_{22} - Q_{12} - 2Q_{66})c^3s,$$

$$\overline{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})s^2c^2 + Q_{66}(s^4 + c^4)$$

Transformation of stresses



Stresses in given coordinate system



Principal stresses

$$\sigma_1 = \sigma_x \, cos^2\,\theta + \sigma_y \, sin^2\,\theta + 2\,\tau_{xy} sin(\theta)\,cos(\theta)$$

$$\tau_{12} = \sigma_x \sin(\theta)\cos(\theta) - \sigma_y \sin(\theta)\cos(\theta) + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

Stress Transformation

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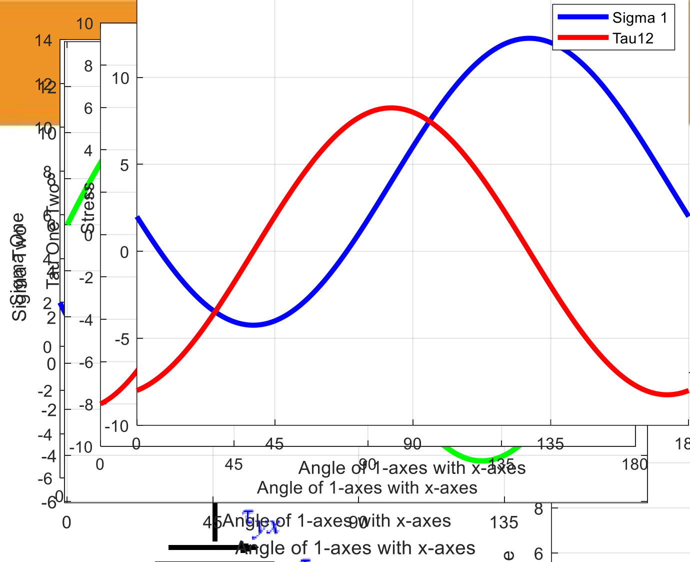
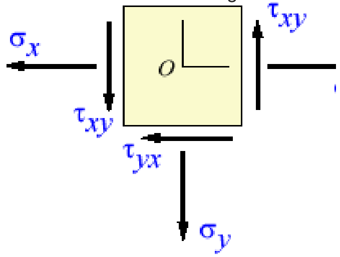
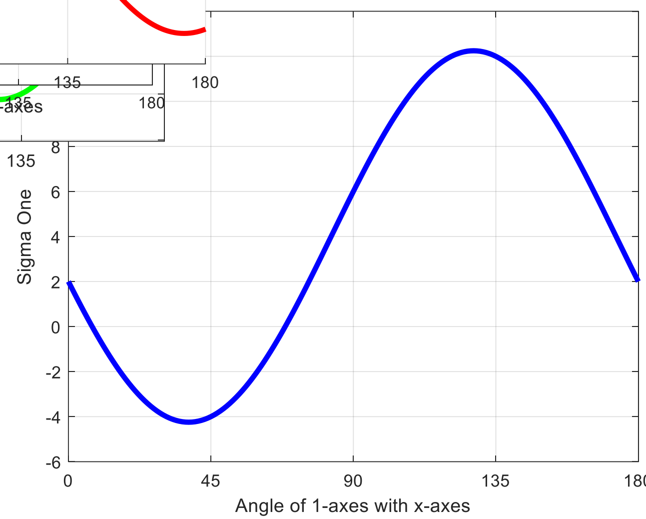
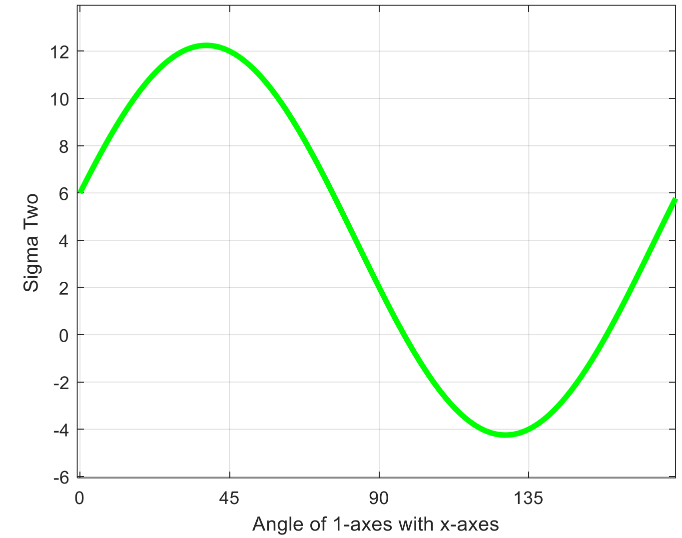
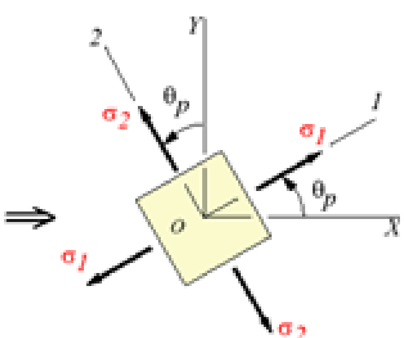




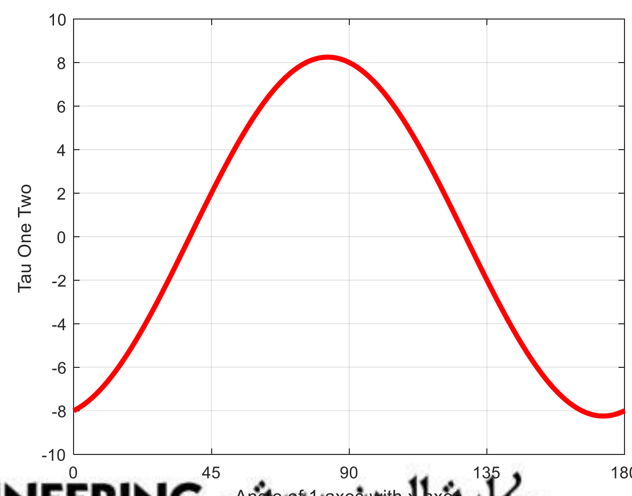
Stresses in given coordinate system

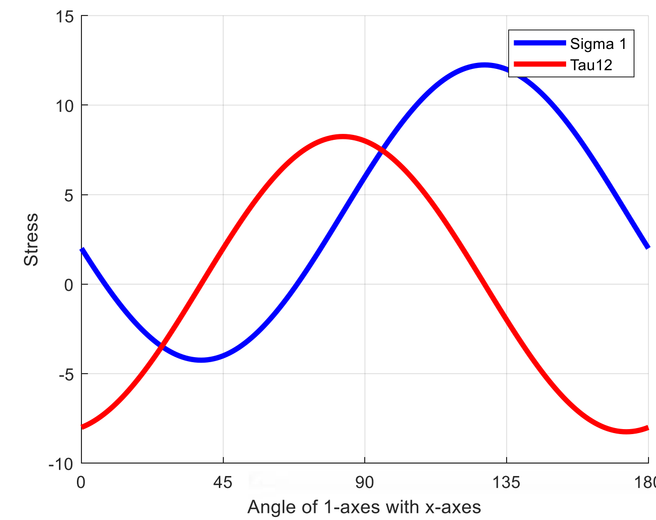






Principal stresses





Example 3.1Structural Design Challenge

You are an engineer designing a lightweight aerospace panel. The material used is a unidirectional composite lamina subjected to normal and shear stresses.

- How does stress transformation affect material behavior?
- How would a 45° rotation impact the stress distribution?

Given Data:

A **unidirectional composite lamina** is subjected to the following stresses:

- Normal stress in the **x-direction**: $\sigma_x = 50$ MPa
- Normal stress in the **y-direction**: $\sigma_y = 30$ MPa
- Shear stress: $\tau_{xy} = 10$ MPa
- The lamina is rotated by **45°**.

We need to find the **transformed stresses** (σ'_x, σ'_y, and τ'_{xy}) in the new coordinate system.

Solution:

$$\sigma'_x = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\sigma'_y = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta$$

$$\tau'_{xy} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

For $\theta = 45^\circ$:

- $\cos 90^\circ = 0$
- $\sin 90^\circ = 1$

Using the given values ($\sigma_x = 50$, $\sigma_y = 30$, $\tau_{xy} = 10$):

$$\sigma'_x = \frac{50+30}{2} + \frac{50-30}{2} \cdot 0 + (10 \times 1)$$

$$\sigma'_x = \frac{80}{2} + 0 + 10 = 40 + 10 = 50 \text{ MPa}$$

$$\sigma'_y = \frac{50+30}{2} - \frac{50-30}{2} \cdot 0 - (10 \times 1)$$

$$\sigma'_y = \frac{80}{2} - 0 - 10 = 40 - 10 = 30 \text{ MPa}$$

$$\tau'_{xy} = -\frac{50-30}{2} \cdot 1 + (10 \times 0)$$

$$\tau'_{xy} = -\frac{20}{2} + 0 = -10 \text{ MPa}$$

Example 3.2

You are analyzing the structural integrity of a Graphite/Epoxy lamina under multi-axial loading conditions.

How can we determine if a lamina will fail under given stresses?

Which failure theories should be used?

Using Maximum Stress Failure Theory:

- Failure occurs if any of these conditions are violated:

$$\sigma_1 < X_t, \quad \sigma_1 > X_c$$

$$\sigma_2 < Y_t, \quad \sigma_2 > Y_c$$

$$\tau_{12} < S$$

Evaluating each condition:

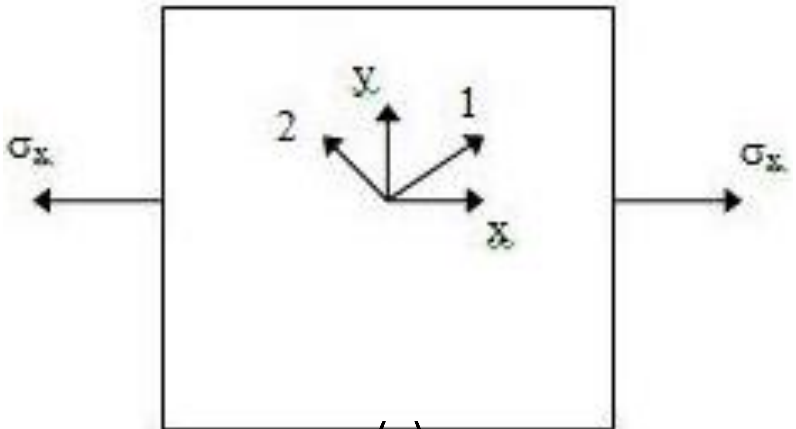
- $100 < 200$ → Safe
- $30 < 50$ → Safe
- $20 < 40$ → Safe

Result:

- The lamina is safe under the given stress conditions.

Pure Axial Load in Direction x

$$\sigma_x \neq 0, \sigma_y = 0, \tau_{xy} = 0$$



(a)

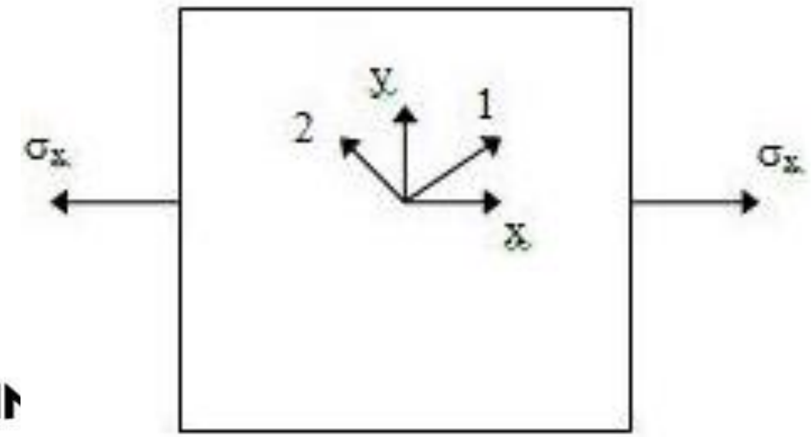
$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \cancelto{0}{\sigma_y} \\ \cancelto{0}{\tau_{xy}} \end{bmatrix}$$

FIGURE 3.3
Application of stresses to find engineering constants of an angle lamina

Engineering Constants

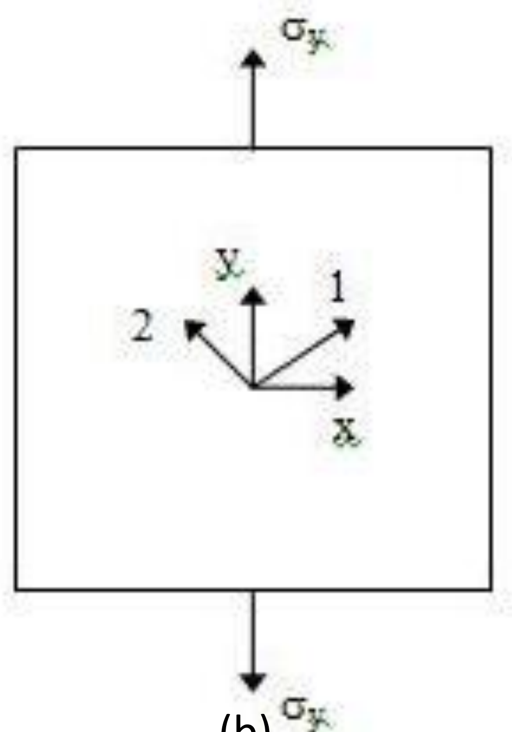
$$\varepsilon_x = \bar{S}_{11}\sigma_x \qquad \varepsilon_y = \bar{S}_{12}\sigma_x \qquad \gamma_{xy} = \bar{S}_{16}\sigma_x$$

$$E_x \equiv \frac{\sigma_x}{\varepsilon_x} = \frac{1}{\bar{S}_{11}} \qquad \nu_{xy} \equiv -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\bar{S}_{12}}{\bar{S}_{11}} \qquad \frac{1}{m_x} \equiv -\frac{\sigma_x}{\gamma_{xy}E_1} = -\frac{1}{\bar{S}_{16}E_1}$$



Pure Axial Load in Direction y

$$\sigma_x = 0, \sigma_y \neq 0, \tau_{xy} = 0$$



(b)

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} \cancelto{0}{\sigma_x} \\ \sigma_y \\ \cancelto{0}{\tau_{xy}} \end{bmatrix}$$

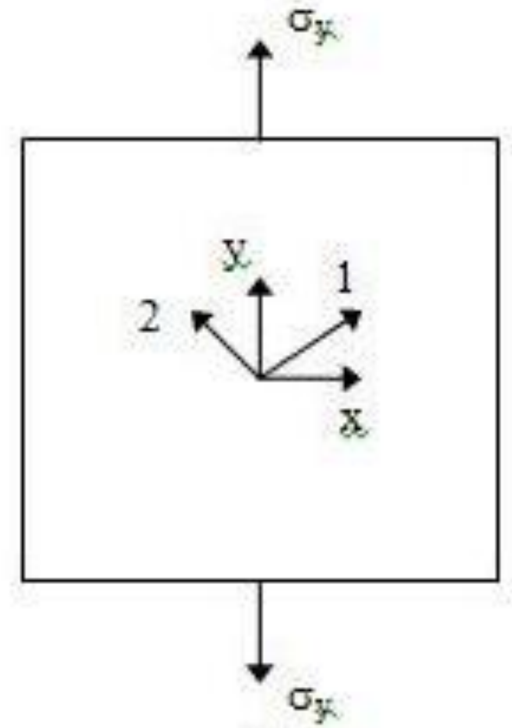
FIGURE 3.4
Application of stresses to find engineering constants of an angle lamina

Engineering Constants

$$\varepsilon_x = \bar{S}_{12}\sigma_y \qquad \varepsilon_y = \bar{S}_{22}\sigma_y \qquad \gamma_{xy} = \bar{S}_{26}\sigma_y$$

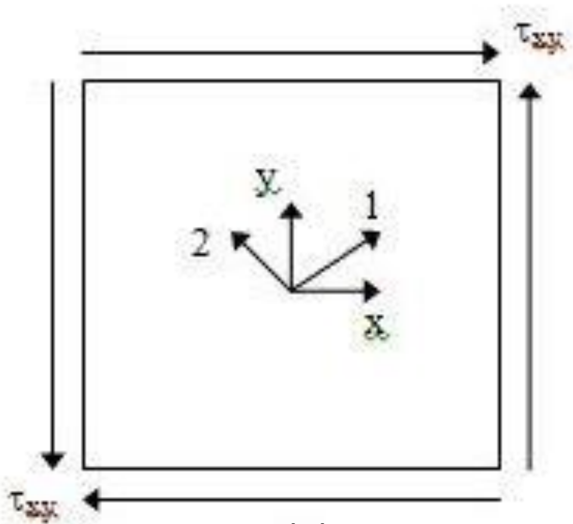
$$\nu_{yx} \equiv -\frac{\varepsilon_x}{\varepsilon_y} = -\frac{\bar{S}_{12}}{\bar{S}_{22}} \qquad E_y \equiv \frac{\sigma_y}{\varepsilon_y} = \frac{1}{\bar{S}_{22}} \qquad \frac{1}{m_y} \equiv -\frac{\sigma_y}{\gamma_{xy}E_1} = -\frac{1}{\bar{S}_{26}E_1}$$

$$\frac{\nu_{yx}}{E_y} = \frac{\nu_{xy}}{E_x}$$



Pure Shear Load in x-y Plane

$$\sigma_x = 0, \sigma_y = 0, \tau_{xy} \neq 0$$



(c)

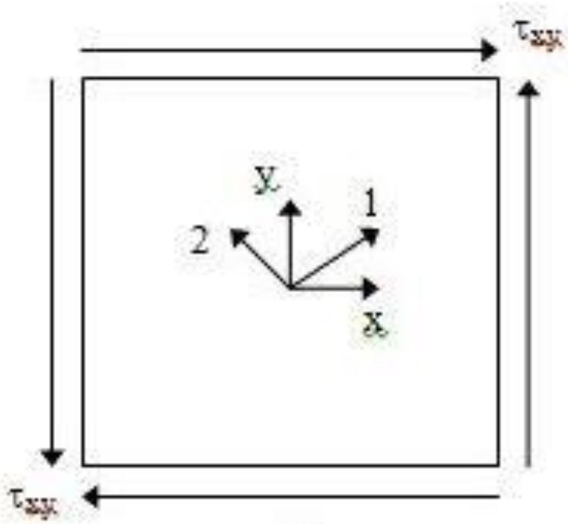
$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \nearrow 0 \\ \sigma_y \nearrow 0 \\ \tau_{xy} \end{bmatrix}$$

FIGURE 3.5
Application of stresses to find engineering constants of an angle lamina

Engineering Constants

$$\varepsilon_x = \bar{S}_{16}\,\tau_{xy} \qquad \varepsilon_y = \bar{S}_{26}\,\tau_{xy} \qquad \gamma_{xy} = \bar{S}_{66}\,\tau_{xy}$$

$$\frac{1}{m_x} = -\frac{\tau_{xy}}{\varepsilon_x E_1} = -\frac{1}{\bar{S}_{16} E_1} \qquad \frac{1}{m_y} = -\frac{\tau_{xy}}{\varepsilon_y E_1} = -\frac{1}{\bar{S}_{26} E_1} \qquad G_{xy} \equiv \frac{\tau_{xy}}{\gamma_{xy}} = \frac{1}{\bar{S}_{66}}$$



Stress-Strain Relationships for Angle Lamina

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{v_{xy}}{E_x} & -\frac{m_x}{E_1} \\ -\frac{v_{xy}}{E_x} & \frac{1}{E_y} & -\frac{m_y}{E_1} \\ -\frac{m_x}{E_1} & -\frac{m_y}{E_1} & \frac{1}{G_{xy}} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

Engineering Constant E_x of an Angle Lamina

$$\frac{1}{E_x} = \bar{S}_{11}$$

$$= S_{11} c^4 + \left(2 S_{12} + S_{66}\right) s^2 c^2 + S_{22} s^4$$

$$= \frac{1}{E_1} c^4 + \left(\frac{1}{G_{12}} - \frac{2 v_{12}}{E_1}\right) s^2 c^2 + \frac{1}{E_2} s^4$$

Engineering Constant E_y of an Angle Lamina

$$\frac{1}{E_y} = \bar{S}_{22}$$

$$= S_{11} s^4 + (2S_{12} + S_{66}) c^2 s^2 + S_{22} c^4$$

$$= \frac{1}{E_1} s^4 + \left(-\frac{2\nu_{12}}{E_1} + \frac{1}{G_{12}} \right) c^2 s^2 + \frac{1}{E_2} c^4$$

Engineering Constant G_{xy} of an Angle Lamina

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$$\frac{1}{G_{xy}} = \bar{S}_{66}$$

$$= 2\left(2S_{11} + 2S_{22} - 4S_{12} - S_{66}\right)s^2c^2 + S_{66}\left(s^4 + c^4\right)$$

$$= 2\left(\frac{2}{E_1} + \frac{2}{E_2} + \frac{4v_{12}}{E_1} - \frac{1}{G_{12}}\right)s^2c^2 + \frac{1}{G_{12}}\left(s^4 + c^4\right)$$

Engineering Constant m_x of an Angle Lamina

$$m_x = -\bar{S}_{16} E_1$$

$$= -E_1\left[\left(2S_{11} - 2S_{12} - S_{66}\right)sc^3 - \left(2S_{22} - 2S_{12} - S_{66}\right)s^3c\right]$$

$$= E_1\left[\left(-\frac{2}{E_1} - \frac{2\nu_{12}}{E_1} + \frac{1}{G_{12}}\right)s\,c^3 + \left(\frac{2}{E_2} + \frac{2\nu_{12}}{E_1} - \frac{1}{G_{12}}\right)s^3c\right]$$

Engineering Constant m_y of an Angle Lamina

$$m_y = -\bar{S}_{26} E_1$$

$$= -E_1\left[\left(2S_{11} - 2S_{12} - S_{66}\right)s^3 c - \left(2S_{22} - 2S_{12} - S_{66}\right)s\,c^3\right]$$

$$= E_1\left[\left(-\frac{2}{E_1} - \frac{2\nu_{12}}{E_1} + \frac{1}{G_{12}}\right)s^3 c + \left(\frac{2}{E_2} + \frac{2\nu_{12}}{E_1} - \frac{1}{G_{12}}\right)s\,c^3\right]$$

Example 3.3 Optimizing Engineering Constants

You are tasked with selecting a composite lamina for an automotive application requiring high stiffness and minimal weight.

- What role do engineering constants play in material selection?
- How can compliance and stiffness matrices be used for material evaluation?

A composite lamina with the following properties:

- Longitudinal modulus: $E_1 = 140$ GPa
- Transverse modulus: $E_2 = 10$ GPa
- Shear modulus: $G_{12} = 5$ GPa
- Poisson's ratio: $\nu_{12} = 0.3$

Solution

Using stress transformation equations:

$$\sigma'_x = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma'_y = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\tau'_{xy} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

Substituting values for θ = 45°:

- $\sigma'_x = 40$ MPa
- $\sigma'_y = 20$ MPa
- $\tau'_{xy} = 15$ MPa

Invariant Form of Stiffness Matrix

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix},$$

$$\overline{Q}_{11} = U_1 + U_2 \cos 2\theta + U_3 \cos 4\theta$$

$$\overline{Q}_{12} = U_4 - U_3 \cos 4\theta$$

$$\overline{Q}_{22} = U_1 - U_2 \cos 2\theta + U_3 \cos 4\theta$$

$$\overline{Q}_{16} = \frac{U_2}{2} \sin 2\theta + U_3 \sin 4\theta$$

$$\overline{Q}_{26} = \frac{U_2}{2} \sin 2\theta - U_3 \sin 4\theta$$

$$\overline{Q}_{66} = \frac{1}{2}(U_1 - U_4) - U_3 \cos 4\theta$$

$$U_1 = \frac{1}{8}(3Q_{11} + 3Q_{22} + 2Q_{12} + 4Q_{66})$$

$$U_2 = \frac{1}{2}(Q_{11} - Q_{22})$$

$$U_3 = \frac{1}{8}(Q_{11} + Q_{22} - 2Q_{12} - 4Q_{66})$$

$$U_4 = \frac{1}{8}(Q_{11} + Q_{22} + 6Q_{12} - 4Q_{66})$$

Invariant Form of Compliance Matrix

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix},$$

$$\bar{S}_{11} = V_1 + V_2 \cos 2\theta + V_3 \cos 4\theta,$$

$$\bar{S}_{12} = V_4 - V_3 \operatorname{Cos} 4\theta,$$

$$\bar{S}_{22} = V_1 - V_2 \operatorname{Cos} 2\theta + V_3 \operatorname{Cos} 4\theta,$$

$$\bar{S}_{16} = V_2 \sin 2\theta + 2V_3 \sin 4\theta,$$

$$\bar{S}_{26} = V_2 \operatorname{Sin} 2\theta - 2V_3 \operatorname{Sin} 4\theta,$$

$$\bar{S}_{66} = 2(V_1 - V_4) - 4V_3 \operatorname{Cos} 4\theta,$$

$$V_1 = \frac{1}{8}(3S_{11} + 3S_{22} + 2S_{12} + S_{66}),$$

$$V_2 = \frac{1}{2}(S_{11} - S_{22}),$$

$$V_3 = \frac{1}{8}(S_{11} + S_{22} - 2S_{12} - S_{66}),$$

$$V_4 = \frac{1}{8}(S_{11} + S_{22} + 6S_{12} - S_{66})$$

Strength Failure Theories for an Angle Lamina

- The failure theories are generally based on the normal and shear strengths of a unidirectional lamina.

- In the case of a unidirectional lamina, the five strength parameters are:

  - ❑ Longitudinal tensile strength $(\sigma_1^T)_{ult}$
  - ❑ Longitudinal compressive strength $(\sigma_1^C)_{ult}$
  - ❑ Transverse tensile strength $(\sigma_2^T)_{ult}$
  - ❑ Transverse compressive strength $(\sigma_2^C)_{ult}$
  - ❑ In-plane shear strength $(\tau_{12})_{ult}$

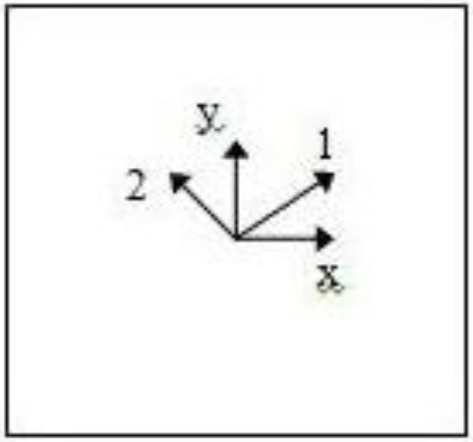
Maximum Stress Failure Theory

- The lamina is considered to be failed if:

$$-\left(\sigma_1^C\right)_{ult} \le \sigma_1 \le \left(\sigma_1^T\right)_{ult}, \quad or$$

$$-\left(\sigma_2^C\right)_{ult} \le \sigma_2 \le \left(\sigma_2^T\right)_{ult}, \quad or$$

$$-\left(\tau_{12}\right)_{ult} \le \tau_{12} \le \left(\tau_{12}\right)_{ult}$$



is violated.

- Note that all five strength parameters are positive numbers.
- Each component of stress does not interact with each other.

Example 3.4

Find the maximum value of S>0 if a stress of $\sigma_x = 2S, \sigma_y = -3S, \text{and } \tau_{xy} = 4S$ is applied to a 60° lamina of Graphite/Epoxy. Use Maximum Stress failure theory. Use properties of a unidirectional Graphite/Epoxy lamina given in Table 2.1 of the textbook Mechanics of Composite Materials by Autar Kaw.

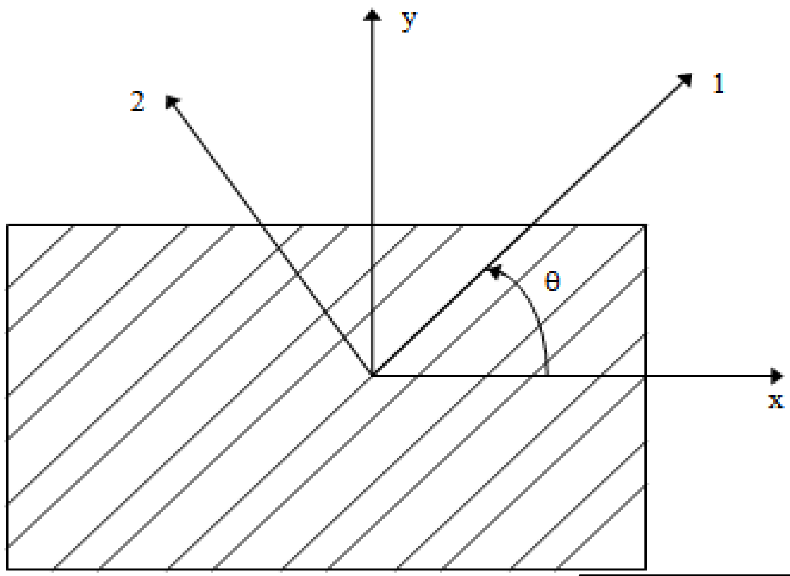


FIGURE 3.5
Off-axis loading in the x-direction

Solution

The stresses in the local axes are

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} 0.2500 & 0.7500 & 0.8660 \\ 0.7500 & 0.2500 & -0.8660 \\ -0.4330 & 0.4330 & -0.5000 \end{bmatrix} \begin{bmatrix} 2S \\ -3S \\ 4S \end{bmatrix}$$

$$= \begin{bmatrix} 0.1714 \times 10^1 \\ -0.2714 \times 10^1 \\ -0.4165 \times 10^1 \end{bmatrix} S.$$

Solution

The ultimate strengths of a unidirectional Graphite/Epoxy lamina are:

$$\left(\sigma_1^T\right)_{ult} = 1500 \text{ MPa},$$

$$\left(\sigma_1^C\right)_{ult} = 1500 \text{ MPa},$$

$$\left(\sigma_2^T\right)_{ult} = 40 \text{ MPa},$$

$$\left(\sigma_2^C\right)_{ult} = 246 \text{ MPa},$$

$$\left(\tau_{12}\right)_{ult} = 68 \text{ MPa}.$$

Solution

Then using the inequalities of the Maximum Stress Failure Theory:

$$-1500(10^6) < 0.1714(10^1)S < 1500(10^6),$$
$$-246(10^6) < -0.2714(10^1)S < 40(10^6),$$
$$-68(10^6) < -0.4165(10^1)S < 68(10^6),$$

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} 0.1714 \times 10^1 \\ -0.2714 \times 10^1 \\ -0.4165 \times 10^1 \end{bmatrix} S.$$

or, $$-875.1(10^6) < S < 875.1(10^6),$$
$$-14.73(10^6) < S < 90.64(10^6)$$
$$-16.33(10^6) < S < 16.33(10^6).$$

All the inequality conditions (and $S > 0$)are satisfied if $0 < S < 16.33\,\text{MPa}.$
The above inequalities also show that the angle lamina will fail in shear. The maximum stress that can be applied before failure is:

$$\sigma_x = 32.66\ MPa, \sigma_x = -48.99\ MPa, \tau_{xy} = 65.32\ MPa.$$